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How to Determine the Capital Requirement for a Portfolio of Annuity Liabilities

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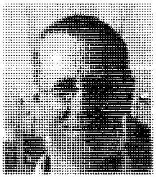

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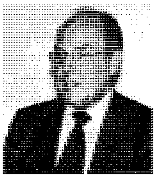

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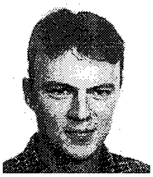

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ABSTRACT

This paper illustrates an analytic method that can be used to determine the total capital requirements necessary to properly provide for the future obligations of a portfolio of annuity liabilities and to protect the enterprise from the related risks it faces. This example is based on the work of Kaas, Dhaene and Goovaerts (2000).

I. INTRODUCTION

The determination of these requirements entails the analysis of the distribution function, more specifically the tail of the distribution function (or the catastrophic part) of the present value of the future cash flows.

The projection of the future cash flows relating to the annuity obligations, and the subsequent determination of their present value, may be at its simplest, when the amount and timing of the asset and liability cash flows are insensitive to varying economic conditions. Even in these circumstances, the possible impact of mortality improvements and the future course of the reinvestment assumptions are important. On the other hand, these computations are most complex, when the timing and amount of the cash flows are affected by the economic scenario (e.g. annuity payments whose amount and timing are solely, or partly, driven by economic performance of some type as well as to the policyholder reaction to such performance).

The use of inappropriately simple (i.e. perhaps for computational ease or convenience) methods for these computations, is likely to introduce hidden "surplus" (could be a deficit too). Capital requirements calculated under such a regime may not reflect the true capital required to support the insurer's business.

In order to compute the likelihood that an insurer will not be able to meet its obligations when they fall due, knowledge is required of the nature of the cash flows and any underlying stochastic process which drives their timing and amount. The method illustrated in this example starts from this knowledge. It enables the actuary to determine (approximately) the relevant probabilities of extreme events. It allows the actuary to determine the necessary provision, based on the level safety desired.

II. THE CASH FLOW OF THE FUTURE LIABILITIES

Firstly, the cash flows depicting the future payments of the annuity portfolio are projected into the future. The cash flows represent, for each year between 2002 and 2079, the expected payment of that year, adjusted with a safety margin. The expectations of the payments due are calculated using realistic technical bases concerning disability, morbidity etc. The additional margin is a safety margin against the possible negative deviations and also includes costs.

III. THE DISTRIBUTION OF THE CAPITAL REQUIREMENT

We will determine the present value at January 1, 2001, which we will consider as the time 0. The time unit is chosen to be one year.

Let α_i be the amount due at time i. The stochastic capital requirement, V, is then given by:

$$V = \alpha_1 e^{-Y_1} + \alpha_2 e^{-(Y_1+Y_2)} + \ldots + \alpha_n e^{-(Y_1+Y_2+\ldots+Y_n)}$$

where Y_1 is the return on the investments from 2001 to 2002, Y_2 is the return from 2002 to 2003, etc.

We will assume that the yearly returns Y_i are normally distributed with mean μ and variance σ^2. We also assume that the yearly returns are mutually independent.

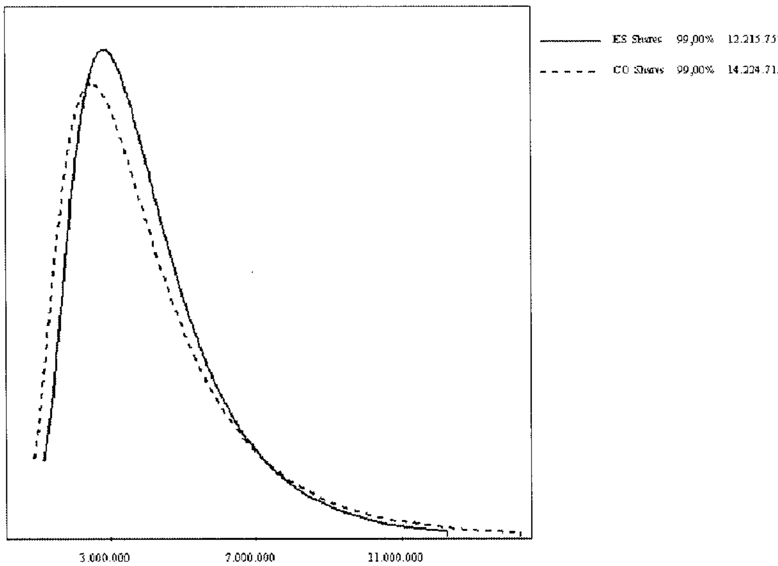
In the following Table, a number of possible investment strategies with values of the parameters μ and σ are given (figures fictious, for illustrative purposes only).

	μ	σ
100% Shares	0.10	0.15
75% / 25%	0.09	0.12295
50% / 50%	0.08	0.09922
25% / 75%	0.07	0.08173
100% Bonds	0.06	0.075

It is clear that V is a sum of dependent random variables. Indeed, the different discount factors will be strongly positive dependent. The computation of the distribution function of V cannot be performed exactly. One could try to determine the distribution function of V by simulation, but this will lead to untrustable estimates for the tail probabilities. Moreover, simulation will be very time-consuming for determining the optimal asset mix. Recent actuarial research results allow to compute lower and upper bounds for the distribution of V. It will be shown that these lower and upper bounds are often close to each other, which of course illustrates the accurateness of the approximations.

On the next pages one finds upper and lower bounds for the distribution of V for the different investment strategies. These bounds are upper and lower bounds in the sense of convex order for the exact but unknown distribution function. The upper bound is denoted by CO (dotted line), the lower bound by ES (full line). We also present approximations for the percentile, the expected shortfall and the conditional tail expectation at different levels.

A. *100% Shares*

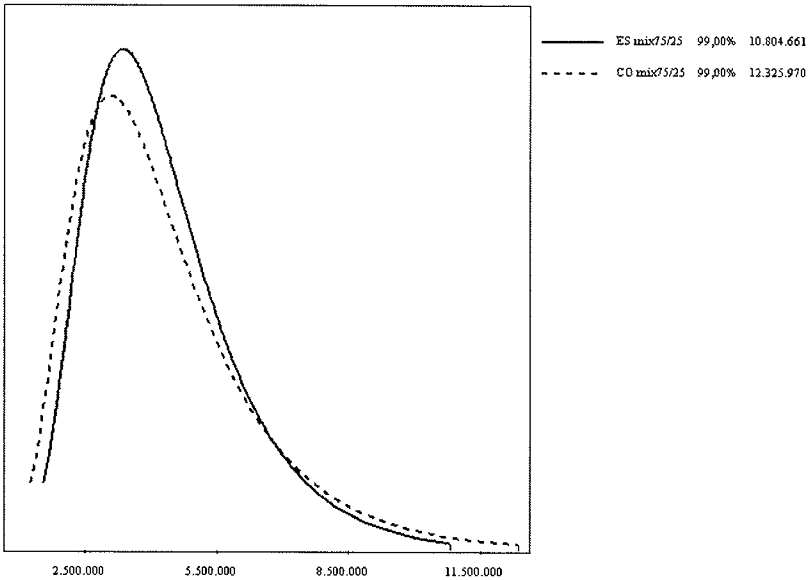


	Lower bound	Upper bound	Exact
Mean	4.225.360	4.225.360	4.225.360
StDev	2.253.463	2.721.836	2.274.370

Lower bound	Percentile	Expected Shortfall	Conditional Tail Expectation
80%	5.579.746	428.101	7.720.253
90%	6.988.839	225.126	9.240.099
95%	8.455.795	119.612	10.848.030
99%	12.215.757	27.9311	15.008.871

Upper bound	Percentile	Expected Shortfall	Conditional Tail Expectation
80%	5.712.519	537.425	8.399.644
90%	7.421.786	291.727	10.339.051
95%	9.263.053	159.533	12.453.716
99%	14.224.715	39.578	18.182.529

B. *75% Shares / 25% Bonds*

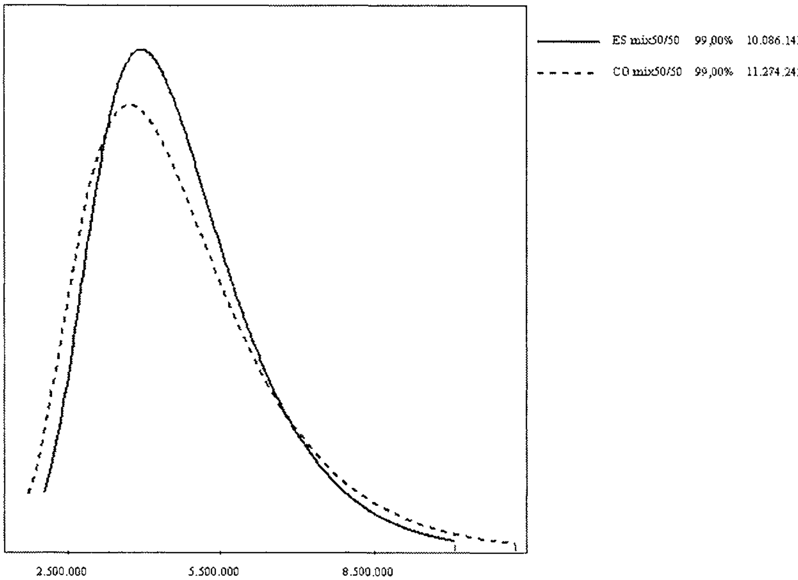


	Lower bound	Upper bound	Exact
Mean	4.413.501	4.413.501	4.413.501
StDev	1.898.995	2.267.604	1.908.791

Lower bound	Percentile	Expected Shortfall	Conditional Tail Expectation
80%	5.667.668	339.903	7.367.182
90%	6.827.381	172.437	8.551.752
95%	7.987.992	88.783	9.763.652
99%	10.804.661	19.474	12.752.102

Upper bound	Percentile	Expected Shortfall	Conditional Tail Expectation
80%	5.820.015	422.885	7.934.440
90%	7.226.903	220.069	9.427.594
95%	8.673.925	115.924	10.992.403
99%	12.325.970	26.654	14.991.412

C. 50% Shares / 50% Bonds

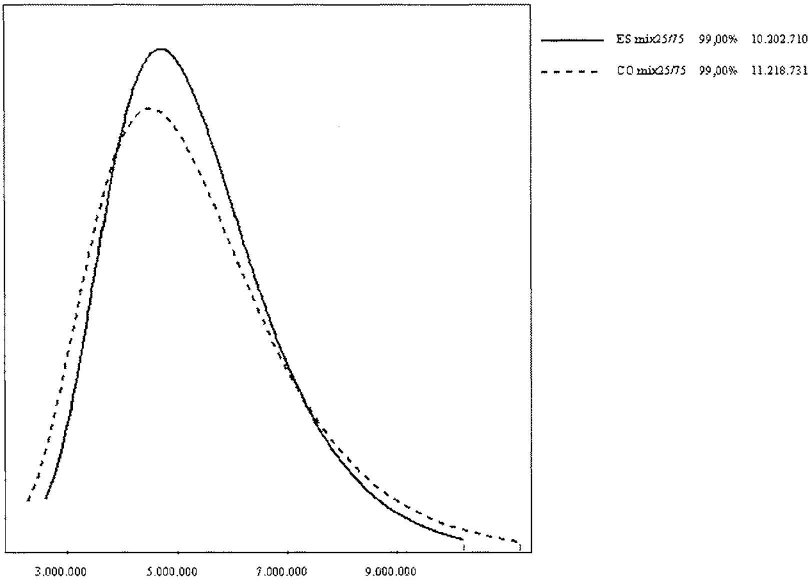


	Lower bound	Upper bound	Exact
Mean	4.783.853	4.783.853	4.783.853
StDev	1.660.798	1.966.517	1.665.608

Lower bound	Percentile	Expected Shortfall	Conditional Tail Expectation
80%	5.958.488	279.343	7.355.204
90%	6.940.005	137.304	8.313.047
95%	7.889.057	68.775	9.264.548
99%	10.086.143	14.284	11.514.564

Upper bound	Percentile	Expected Shortfall	Conditional Tail Expectation
80%	6.118.644	343.966	7.838.475
90%	7.305.082	172.501	9.030.097
95%	8.477.273	87.956	10.236.398
99%	11.274.242	18.941	13.168.379

D. *25% Shares / 75% Bonds*

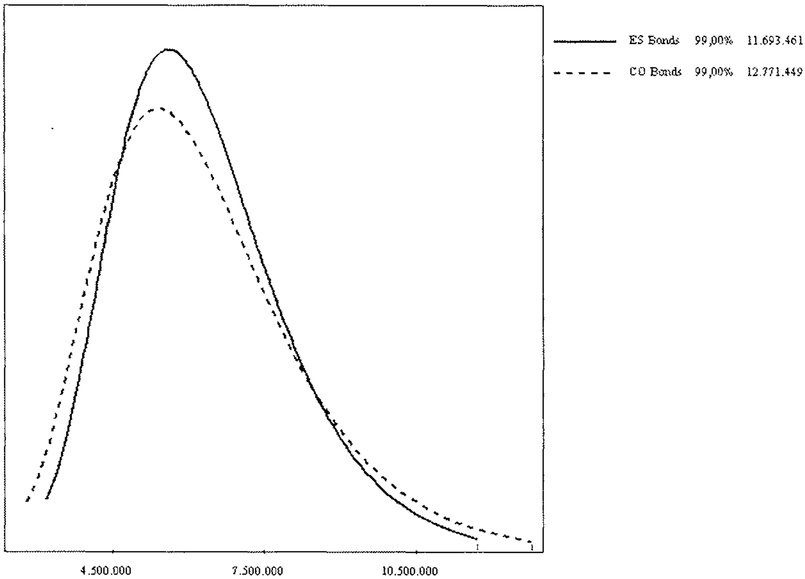


	Lower bound	Upper bound	Exact
Mean	5.399.304	5.399.304	5.399.304
StDev	1.568.746	1.845.610	1.571.544

Lower bound	Percentile	Expected Shortfall	Conditional Tail Expectation
80%	6.556.404	250.846	7.810.633
90%	7.455.679	120.507	8.660.749
95%	8.303.957	59.174	9.487.444
99%	10.202.710	11.818	11.384.461

Upper bound	Percentile	Expected Shortfall	Conditional Tail Expectation
80%	6.722.113	305.781	8.251.018
90%	7.802.926	149.299	9.295.919
95%	8.840.306	74.363	10.327.564
99%	11.218.731	15.282	12.746.884

E. *100% Bonds*



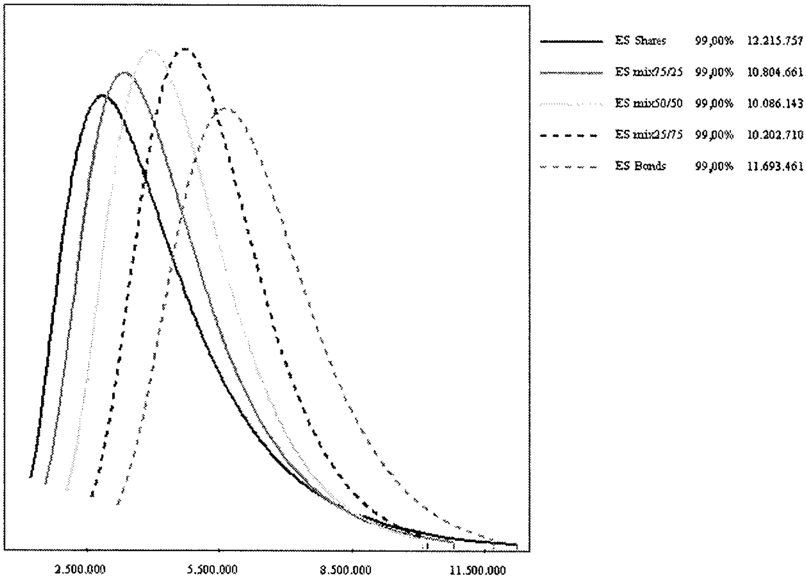
	Lower bound	Upper bound	Exact
Mean	6.379.328	6.379.328	6.379.328
StDev	1.757.507	2.057.314	1.760.055

Lower bound	Percentile	Expected Shortfall	Conditional Tail Expectation
80%	7.690.022	276.730	9.073.672
90%	8.687.722	132.061	10.008.333
95%	9.622.142	64.475	10.911.638
99%	11.693.461	12.730	12.966.449

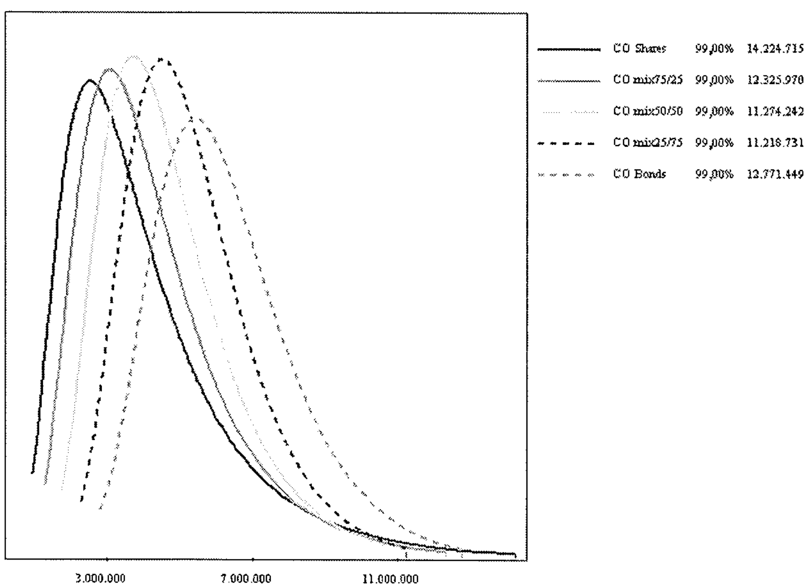
Upper bound	Percentile	Expected Shortfall	Conditional Tail Expectation
80%	7.876.111	334.863	9.550.426
90%	9.068.340	162.155	10.689.894
95%	10.202.571	80.184	11.806.261
99%	12.771.449	16.241	14.395.547

IV. COMPARISON BETWEEN THE DIFFERENT INVESTMENT STRATEGIES

A. *Upper Bound (CO)*



B. *Lower Bounds (ES)*



C. *Remarks*

The expectations of the lower and upper bound of V are always equal. This is because the bounds have exact expectations.

From numerical comparisons it follows that the best estimate for the distribution of the present value of the cash flow under consideration is the lower bound. This is confirmed by the fact that the exact Standard Deviation and the Standard Deviation of the lower bound are very close to each other.

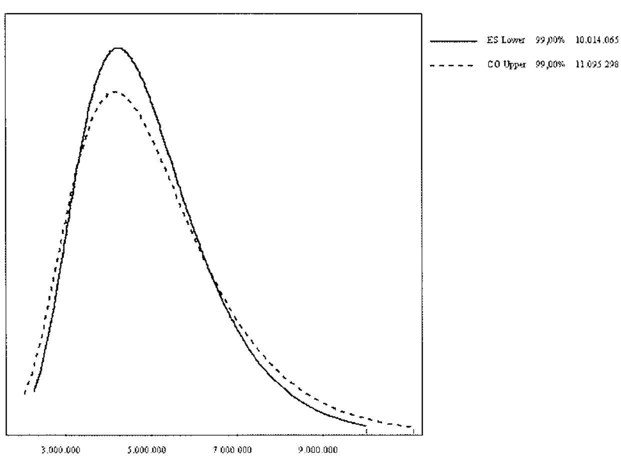
V. THE OPTIMAL ASSET ALLOCATION

The optimal asset allocation is a combination of the different investment strategies (also taking into account the dependencies between the yearly returns), that minimizes a certain criterion.

Here we will assume that the optimal asset allocation minimizes the initial amount that has to be reserved such that, with probability of 99%, all future obligations can be met. Hence, the optimal asset allocation is the one that minimizes the 99%-percentile of V.

Considering the lower bounds, the optimal investment strategy turns out to invest 40,40% in shares and the remaining part in bonds, see the following figure. Considering the upper bounds (which are less accurate) we find that the optimal investment strategy is to invest 35,87% in shares.

A. *Optimal investment strategies*



	Lower bound	Upper bound
Mean	4.986.280	5.095.640
StDev	1.603.779	1.869.963

Lower bound	Percentile	Expected Shortfall	Conditional Tail Expectation
80%	6.142.831	263.904	7.462.353
90%	7.078.531	128.403	8.362.556
95%	7.973.369	63.750	9.248.372
99%	10.014.065	13.011	11.315.205

Upper bound	Percentile	Expected Shortfall	Conditional Tail Expectation
80%	6.410.485	316.332	7.992.144
90%	7.518.701	155.990	9.078.596
95%	8.593.818	78.371	10.161.239
99%	11.095.298	16.386	12.733.889

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